Semester One Examination 2011

Question/answer booklet

MATHEMATICS SPECIALIST
MAS 3C/D
Section Two
(Calculator Assumed)

Student Name: SOLUTIONS

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for section: 100 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section Two. Candidates may use the removable formula sheet from Section One

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler

Special items: drawing instruments, templates, notes on up to two unfolded sheets of A4 paper, and up to three calculators, CAS, graphic or scientific, which satisfy the conditions set by the Curriculum Council for this course.

Important note to candidates

No other items may be taken into the examination room. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor before reading any further.

Structure of this examination

	Number of questions	Working time (minutes)	Marks available
Section One Calculator Free	8	50	40
This Section (Section 2) Calculator Assumed	**13**	**100**	**80**
		Total marks	120

Instructions to candidates

1. The rules for the conduct of WACE external examinations are detailed in the booklet *WACE Examinations Handbook.* Sitting this examination implies that you agree to abide by these rules.

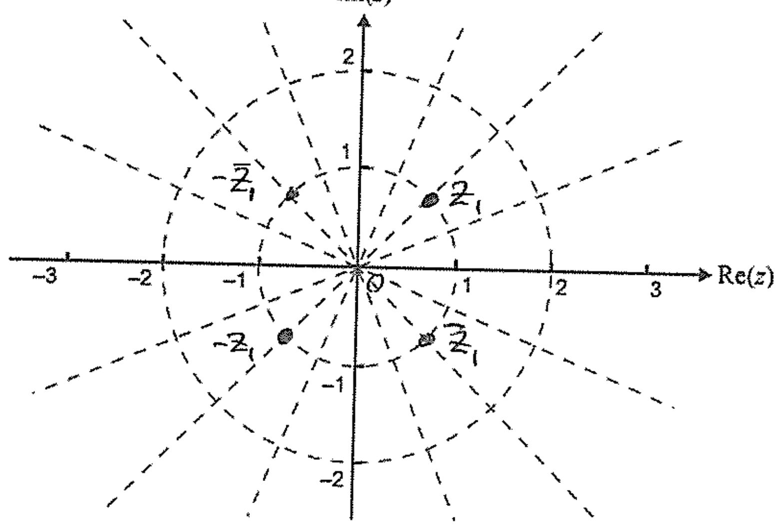
2. Answer the questions in the spaces provided.

3. Show all working clearly. Any question, or part question, worth more than 2 marks requires valid working or justification to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

1. [3 marks]

Let $z_1 = \text{cis}\,(\frac{\pi}{4})$.

(a) Plot and label carefully the points z_1, $-z_1$, $\bar{z}_1$ and $-\bar{z}_1$ on the Argand diagram below. [2]



$z_1, -z_1$ ✓

$\bar{z}_1, -\bar{z}_1$ ✓

(b) Write down the complex equation (in terms of $\bar{z}_1$ and $-\bar{z}_1$) of the straight line which passes through the points z_1 and $-z_1$. [1]

$|z - \bar{z}_1| = |z + \bar{z}_1|$ ✓ (not $z = r\,\text{cis}\,\frac{\pi}{4}$)

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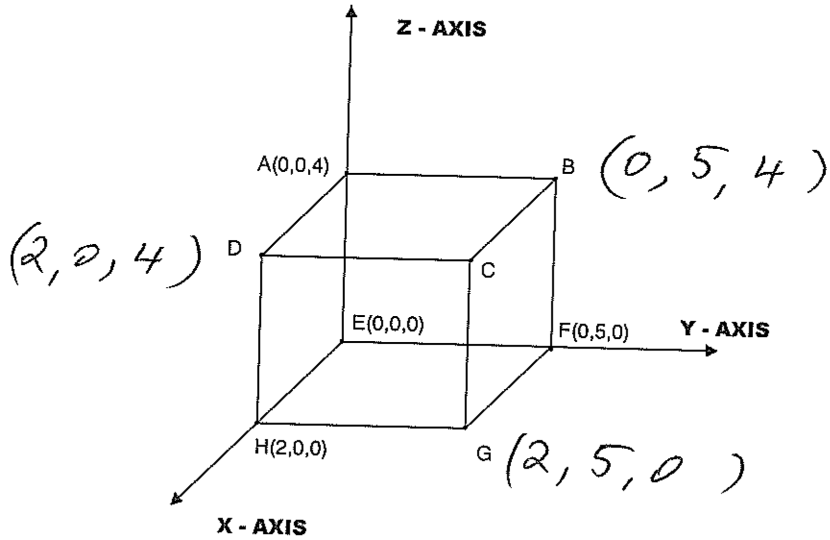
2. [9 marks]



(a) Find the equation of the line parallel to **AG** through the point A. [2]

$$\vec{AG} = \vec{OG} - \vec{OA}$$
$$= \begin{pmatrix}2\\5\\0\end{pmatrix} - \begin{pmatrix}0\\0\\4\end{pmatrix}$$
$$= \begin{pmatrix}2\\5\\-4\end{pmatrix} \checkmark$$

$$\underset{\sim}{r}_{AG} = \begin{pmatrix}0\\0\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\5\\-4\end{pmatrix} \checkmark$$

(b) Find the angle between **GB** and **GF** using vector methods. [3]

$$\vec{GB} = \begin{pmatrix}0\\5\\4\end{pmatrix} - \begin{pmatrix}2\\5\\0\end{pmatrix}$$
$$= \begin{pmatrix}-2\\0\\4\end{pmatrix}$$
$$|\vec{GB}| = \sqrt{20} \checkmark$$

$$\vec{GF} = \begin{pmatrix}0\\5\\0\end{pmatrix} - \begin{pmatrix}2\\5\\0\end{pmatrix}$$
$$= \begin{pmatrix}-2\\0\\0\end{pmatrix}$$
$$|\vec{GF}| = 2 \checkmark$$

$$\vec{GB} \cdot \vec{GF} = 4$$
$$\theta = \cos^{-1}\frac{4}{2\sqrt{20}}$$
$$\theta = 63^\circ \checkmark \quad \text{OR } 1.107^c$$

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(c) Using point A, **AB** and **AD**, write the equation of the plane ABCD. [3]

$\vec{AB} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$ ✓ $\vec{AD} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ ✓

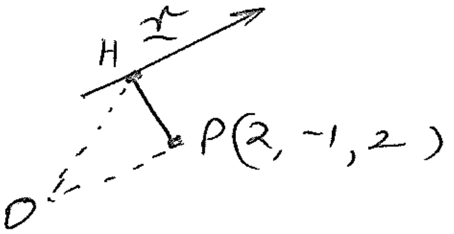
$r_{ABCD} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ ✓

(d) Can the equation of plane ABCD be created using **AB** and **DC**? Explain. [1]

No, $\vec{AB} \parallel \vec{DC}$ ... need two non parallel vectors. ✓

3. [4 marks]

Find the shortest distance between the line $r = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and the point (2, -1, 2).



$\vec{OH} = \begin{pmatrix} -1 + 4t \\ t \\ 7 - 2t \end{pmatrix}$

$\vec{PH} = \vec{OH} - \vec{OP}$ ✓

$= \begin{pmatrix} -3 + 4t \\ 1 + t \\ 5 - 2t \end{pmatrix}$

for shortest distance

$\vec{PH} \cdot \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = 0$ ✓

$4(-3+4t) + 1(1+t) - 2(5-2t) = 0$

$t = 1$ ✓

$\vec{PH} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$|\vec{PH}| = \sqrt{14}$ ✓

(8)

4. [4 marks]

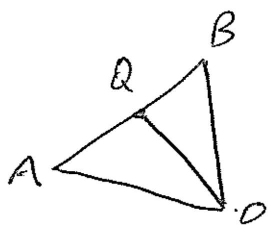
OAB is a triangle with **OA** = 3**i** + 2**j** + $\sqrt{3}$ **k** and **OB** = α **i** where α , which is greater than zero, is chosen so that triangle OAB is isosceles, with $|\mathbf{OA}| = |\mathbf{OB}|$.

(a) Show that $\alpha = 4$. [1]

$|\vec{OA}| = \sqrt{9+4+3} = |\vec{OB}| = 4$

$\Rightarrow \alpha = 4.$

(b) Find **OQ** , where Q is the midpoint of the line segment AB. [1]



$\vec{OQ} = \vec{OA} + \dfrac{\vec{OB} - \vec{OA}}{2}$

$= \frac{7}{2} \underset{\sim}{i} + \underset{\sim}{j} + \frac{\sqrt{3}}{2} \underset{\sim}{k}$

(c) Use a vector method to show that **OQ** is perpendicular to **AB**. [2]

$\vec{AB} = \underset{\sim}{i} - 2\underset{\sim}{j} - \sqrt{3}\,\underset{\sim}{k}$

$\vec{OQ} \cdot \vec{AB} = \frac{7}{2} - 2 - \frac{3}{2}$

$= 0$ for $\vec{OQ} \perp \vec{AB}$

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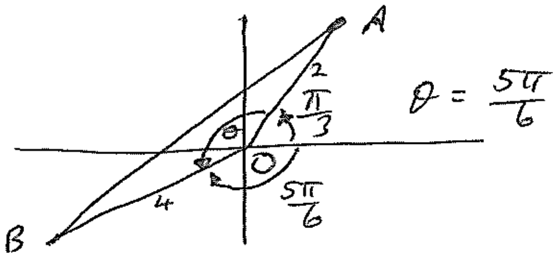
5. [7 marks]

The points A$\left(2, \frac{\pi}{3}\right)$ and B$\left(4, \frac{-5\pi}{6}\right)$ are given in polar co-ordinates.

(a) Find the distance between the two points. [2]



$\theta = \frac{5\pi}{6}$

$d_{AB}^2 = 2^2 + 4^2 - 2 \times 2 \times 4 \times \cos\frac{5\pi}{6}$ ✓

$= 20 + 8\sqrt{3}$

$d_{AB} = 5.819$ ✓

(b) Find two different spirals that contain point A. [2]

$r = k\theta$

$2 = k\left(\frac{\pi}{3}\right)$

$k = \frac{6}{\pi}$

$r = \frac{6}{\pi}\theta$ ✓

OR

$r = k\theta$

$2 = k\left(-\frac{5\pi}{3}\right)$

$k = -\frac{6}{5\pi}$

$r = -\frac{6}{5\pi}\theta$ ✓

OR

$r = k\theta$

$2 = k\left(\frac{7\pi}{3}\right)$

$k = \frac{6}{7\pi}$

$r = \frac{6}{7\pi}\theta$

(c) Find the equation in Polar form and Cartesian form of the **ray** that starts at the origin and contains the point B. [3]

$\theta = -\frac{5\pi}{6}$ ✓ $(r \geq 0)$

$\Rightarrow \frac{y}{x} = \tan^{-1}\frac{5\pi}{6}$

$y = \frac{1}{\sqrt{3}}x, \quad x \leq 0$

✓ ✓

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6. [12 marks]

An aircraft approaching an airport with velocity $\begin{pmatrix} 30 \\ -40 \\ -4 \end{pmatrix}$ metres per second is observed on the control tower radar screen at time t = 0 seconds. Ten seconds later it passes over a navigation beacon with position vector $(-500\mathbf{i} + 2500\mathbf{j})$ metres relative to the base of the control tower, at an altitude of 200 metres.

(a) Show that the position vector of the aircraft relative to the base of the control tower at time t is given by

$\mathbf{r}(t) = (30t - 800)\,\mathbf{i} + (2900 - 40t)\,\mathbf{j} + (240 - 4t)\,\mathbf{k}$ [3]

$$\underset{\sim}{r} = \begin{pmatrix} 30t \\ -40t \\ -4t \end{pmatrix} + \underset{\sim}{c} \rightarrow \begin{pmatrix} -500 \\ 2500 \\ 200 \end{pmatrix} = \begin{pmatrix} 300 \\ -400 \\ -40 \end{pmatrix} + \underset{\sim}{c}$$

$$\underset{\sim}{c} = \begin{pmatrix} -800 \\ 2900 \\ 240 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{r}(t) = (30t - 800)\underset{\sim}{i} + (2900 - 40t)\underset{\sim}{j} + (240 - 4t)\underset{\sim}{k}$$

(b) When does the aircraft land and how far (correct to the nearest metre) from the base of the control tower is the point of landing? [3]

$$240 - 4t = 0 \rightarrow t = 60\text{ s}$$

$$\underset{\sim}{r}(60) = \begin{pmatrix} 1000 \\ 500 \\ 0 \end{pmatrix}$$

$$|\underset{\sim}{r}(60)| = 1118\text{ m}$$

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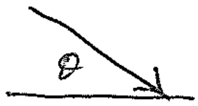
(c) At what angle from the runway, correct to the nearest tenth of a degree, does the aircraft land? [2]

$$\sin\theta = \frac{4}{\sqrt{30^2+40^2+4^2}}$$



$$\theta = 4.6^\circ$$

(d) At what time, correct to the nearest second, is the aircraft closest to the base of the control tower? [2]

$$\underline{r}\cdot\underline{v} = 0 \rightarrow 30(30t-800)-40(2900-4t)-4(240-4t)=0$$

$$\rightarrow t = 56 \text{ sec.}$$

(e) What distance does the aircraft travel from the time it is observed on the radar screen to the time it lands? Give your answer correct to the nearest metre? [2]

$$\text{distance} = \sqrt{30^2+40^2+4^2} \times 60$$

$$= 3010 \text{ m}$$

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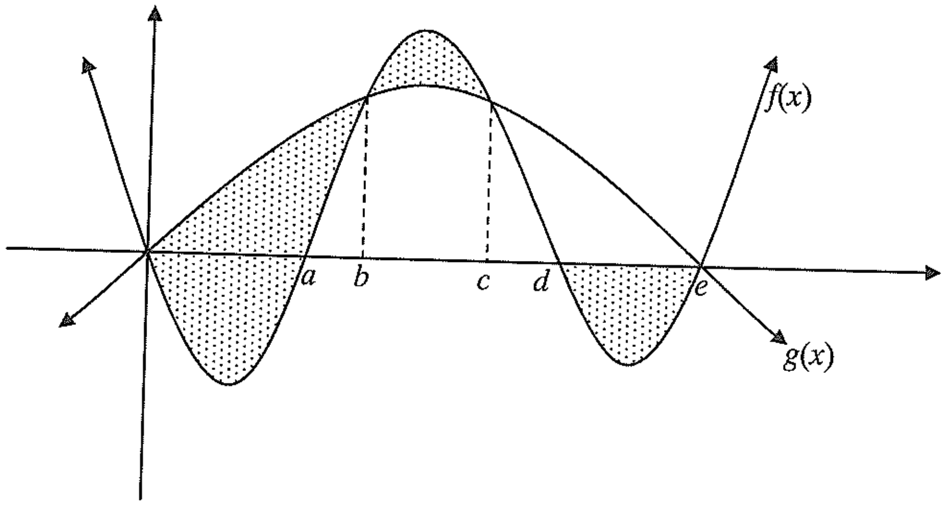
7. [4 marks]

The diagram below shows a sketch of two curves $f(x)$ and $g(x)$.



(a) In terms of $f(x)$, $g(x)$, a, b, c, d and e write an expression that can be used to find the value of the area shade. [2]

$$A = \int_0^b (g(x) - f(x))\,dx + \int_b^c (f(x) - g(x))\,dx - \int_d^e f(x)\,dx$$

(b) Given that $f(x) = x^4 - 10x^3 + 29x^2 - 20x$ and $g(x) = -2x^2 + 10x$ find the value of the shaded area. [2]

$$A = \frac{73}{5} = 14.6$$

where $b = 2$, $c = 3$, $d = 4$ $e = 5$

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8. [7 marks]

Find the equations of *two* of the tangents to the curve $f(x) = \dfrac{kx}{x^2-1}$ that have a gradient of $-k$. Give your answers exactly in terms of k.

$$f'(x) = \frac{k(x^2-1) - 2x(kx)}{(x^2-1)^2} = \frac{-k(x^2+1)}{(x^2-1)^2} \checkmark$$

Since $f'(x) = -k$

$$\frac{-k(x^2+1)}{(x^2-1)^2} = -k \checkmark$$

$$x^2+1 = x^4 - 2x^2 + 1$$

$$x^4 - 3x^2 = 0$$

$$x^2(x^2-3) = 0$$

$$x = 0, \pm\sqrt{3} \checkmark \checkmark$$

if $x = 0$
$y = 0$
$f'(x) = -k$
$y = -kx + c$
$c = 0$
$y = -kx$ ✓

if $x = \sqrt{3}$
$y = \dfrac{k\sqrt{3}}{2}$
$f'(x) = -k$
$y = -kx + c$
$\dfrac{\sqrt{3}}{2}k = -k \cdot \sqrt{3} + c$
$c = \sqrt{3}k + \dfrac{\sqrt{3}k}{2}$
$= \dfrac{3\sqrt{3}}{2}k$
$y = -kx + \dfrac{3\sqrt{3}}{2}k$
✓ ✓

if $x = -\sqrt{3}$
$y = -\dfrac{\sqrt{3}}{2}x$
$f'(x) = -k$
$y = -kx + c$
$-\dfrac{\sqrt{3}}{2}k = \sqrt{3}k + c$
$c = -\sqrt{3}k - \dfrac{\sqrt{3}}{2}k$
$= -\dfrac{3\sqrt{3}}{2}k$
$y = -kx - \dfrac{3\sqrt{3}}{2}k$

ANY TWO

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9. [6 marks]

The total number of bacteria found on a bench top after bleach has been applied can be approximately modelled by the differential equation $\frac{dB}{dt} = -0.1t(B-223)$

where B is the number of bacteria and t is the time measured in seconds.

Three seconds after the bleach was applied it was found that there were 104 000 bacteria still alive on the bench top.

(a) Find B in terms of t. [4]

$$\int \frac{dB}{B-223} = \int -0.1\,t\,dt \quad ✓$$

$$\ln(B-223) = -\frac{t^2}{20} + c \quad ✓ \qquad \text{if } t=3,\ B = 104{,}000$$

$$C = 12 \quad ✓$$

hence $B - 223 = e^{-\frac{t^2}{20}+12}$

$$B = e^{-\frac{t^2}{20}+12} + 223 \quad ✓$$

(b) What was the number of bacteria initially on the bench top? Give your answer to the nearest 1000 bacteria. [1]

at $t = 0$ $\quad B = e^{12} + 223$

$= 162977$

$B = 163{,}000$ ✓

(c) After a long time how many bacteria survived the bleaching process? [1]

as $t \to \infty \quad e^{-\frac{t^2}{20}+12} \to 0$

hence 223 bacteria. ✓

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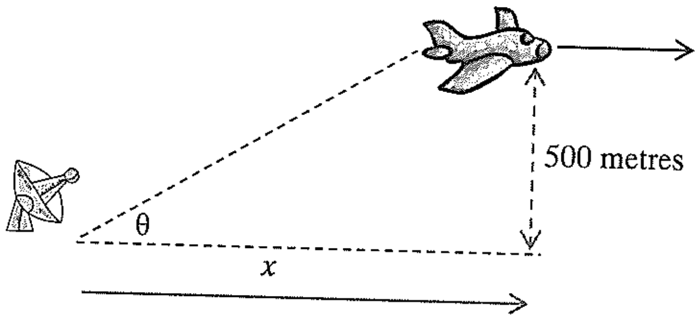
10. [7 marks]

A plane travelling at 720 km/hr at a constant height of 500 metres above ground level is being tracked by radar. The distance along the ground is x and the angle of elevation to the plane is θ, as per the diagram below. At a time $t = 0$ seconds the plane is directly overhead.



$= \frac{720}{3.6}\ ms^{-1}$

$\frac{dx}{dt} = 200\ ms^{-1}$

(a) Find $\frac{d\theta}{dt}$ as a function of θ. [5]

$\tan\theta = \frac{500}{x}$

$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt}$

$= -\frac{\sin^2\theta}{500} \cdot 200$

$\frac{d\theta}{dt} = -\frac{2}{5}\sin^2\theta$

$x = 500(\tan\theta)^{-1}$

$\frac{dx}{d\theta} = -500(\tan\theta)^{-2} \cdot \sec^2\theta$

$\frac{d\theta}{dx} = \frac{-\cos^2\theta\ \tan^2\theta}{500}$

$\frac{d\theta}{dx} = -\frac{\sin^2\theta}{500}$

(b) Calculate $\frac{d\theta}{dt}$ 5 seconds after being overhead. [2]

$\left.\frac{d\theta}{dt}\right|_5 = \left.-\frac{2}{5}\sin^2\theta\right]_{\theta=0.4636}$

at $t=5$, $x = 1000$ m

$\tan\theta = \frac{500}{1000} = \frac{1}{2}$

$\theta = 0.4636^c$

$= -\frac{2}{5}\sin^2 0.4636$

$= 0.08$ rad/sec (clockwise)

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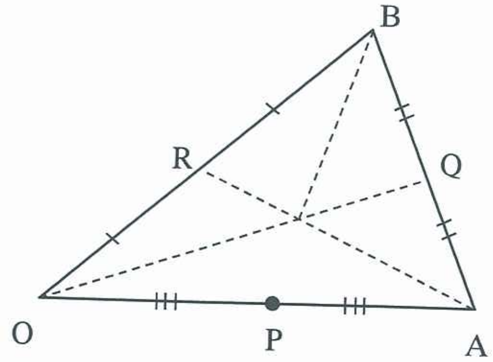
11. [10 marks]

OQ, AR are respectively the medians of Δ OAB (Q and R are the midpoints of AB and OB). Let the medians intersect at M. Also, let

OA = **a**, **OB** = **b**, **OM** = α **OQ** and **AM** = β **AR**.



(a) Find **OQ** and **AR** in terms of **a** and/or **b**. [2]

$\vec{OQ} = b + \frac{1}{2}\vec{BA}$
$= b + \frac{1}{2}(\vec{BO} + \vec{OA})$
$= b + \frac{1}{2}(-b + a)$
$= \frac{1}{2}(a + b)$ ✓

$\vec{AR} = \vec{AO} + \vec{OR}$
$= -a + \frac{1}{2}b$ ✓

(b) Hence, find **OM** and **AM** in terms of **a** and/or **b**. [1]

$\vec{OM} = \alpha\,\vec{OQ}$
$= \frac{\alpha}{2}(a + b)$

$\vec{AM} = \beta\,\vec{AR}$
$= \beta(-a + \frac{1}{2}b)$ ✓

(c) Prove that $\alpha = \beta = \frac{2}{3}$ [3]

Hence using $\vec{OA}$

$\vec{OA} = \vec{OM} + \vec{MA}$

$a = \frac{\alpha}{2}(a + b) - \beta(-a + \frac{1}{2}b)$ ✓
$= \frac{\alpha}{2}a + \frac{\alpha}{2}b + \beta a - \frac{\beta}{2}b$

$\therefore a = a(\frac{\alpha}{2} + \beta) + b(\frac{\alpha - \beta}{2})$

$\Rightarrow \frac{\alpha}{2} + \beta = 1$ and $\frac{\alpha - \beta}{2} = 0$
$\beta = 1 - \frac{\alpha}{2}$ and $\alpha = \beta$ ✓

$\Rightarrow \alpha = 1 - \frac{\alpha}{2}$ ✓
$\frac{3\alpha}{2} = 1$
$\alpha = \frac{2}{3} = \beta$ QED

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(d) Find **BM** and **MP** in terms **a** and/or **b**. [2]

$\vec{BM} = \vec{BO} + \vec{OM}$

$= -\underline{b} + \frac{2}{3} \cdot \frac{1}{2}(\underline{a} + \underline{b})$

$= -\underline{b} + \frac{1}{3}\underline{a} + \frac{1}{3}\underline{b}$

$\vec{BM} = \frac{1}{3}\underline{a} - \frac{2}{3}\underline{b}$ ✓.

$\vec{MP} = \vec{MO} + \vec{OP}$

$= -\frac{1}{3}(\underline{a} + \underline{b}) + \frac{1}{2}\underline{a}$

$= \frac{1}{6}\underline{a} - \frac{1}{3}\underline{b}$ ✓

$\vec{MP} = \frac{1}{6}(\underline{a} - 2\underline{b})$

(e) Hence, prove that B, M and P are collinear. [1]

Collinearity $\Rightarrow \vec{BM} = k\vec{MP}$

$\therefore \frac{1}{3}(\underline{a} - 2\underline{b}) = k \cdot \frac{1}{6}(\underline{a} - 2\underline{b})$

$\Rightarrow k = 2$ ✓ QED

(f) Hence, what can you conclude about the medians of a triangle. [1]

The medians of a Δ intersect at a point ✓ (circumcentre)
-- 2/3 along their respective lengths.

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12. [4 marks]

Use mathematical induction to prove the following identity true for integers $n \geq 1$.

$$\frac{d^{2n}}{dx^{2n}}(\sin x) = (-1)^n \sin x$$

Try $n=1$ $LHS = \frac{d^2}{dx^2}(\sin x) = \frac{d}{dx}(\cos x) = -\sin x$

$RHS = (-1)^1 \sin x$ $\therefore LHS = RHS$ ✓ TRUE

Let $n=k$, assume true $\frac{d^{2k}}{dx^{2k}}(\sin x) = (-1)^k \sin x$

Test $n=k+1$ $LHS = \frac{d^{2(k+1)}}{dx^{2(k+1)}}(\sin x) = \frac{d^2}{dx^2}\left(\frac{d^{2k}}{dx^{2k}}\right)(\sin x)$ ✓

$= \frac{d^2}{dx^2}\left((-1)^k \sin x\right)$ ✓ from $n=k$

$= (-1)^k \cdot -\sin x$ from $n=1$

$= (-1)^{k+1} \sin x$ ✓

$= RHS$ QED.

13. [3 marks]

Show that the solution y = sin x makes the following differential trig identity true for all x.

$$(y^2 \sec^2 x + 1)\left[\sin 2x + 2\frac{dy}{dx}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 1 \qquad \text{for } \sec x = \frac{1}{\cos x}$$

$LHS = \left(\frac{\sin^2 x}{\cos^2 x} + 1\right)(\sin 2x + 2\cos x \cdot -\sin x + \cos^2 x)$ ✓

$= (\tan^2 x + 1)\cos^2 x$ ✓

$= \sec^2 x \cdot \cos^2 x$

$= \frac{1}{\cos^2 x}\cos^2 x$

$= 1$ ✓

$= RHS$ QED

End of Questions

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